Problem 1.63

(a) Find the divergence of the function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r}.$$

First compute it directly, as in Eq. 1.84. Test your result using the divergence theorem, as in Eq. 1.85. Is there a delta function at the origin, as there was for $\hat{\mathbf{r}}/r^2$? What is the general formula for the divergence of $r^n\hat{\mathbf{r}}$? [*Answer:* $\nabla\cdot(r^n\hat{\mathbf{r}}) = (n+2)r^{n-1}$, unless $n=-2$, in which case it is $4\pi\delta^3(\mathbf{r})$; for $n<-2$, the divergence is ill defined at the origin.]

(b) Find the *curl* of $r^n\hat{\mathbf{r}}$. Test your conclusion using Prob. 1.61b. [*Answer:* $\nabla\times(r^n\hat{\mathbf{r}}) = \mathbf{0}$.]

Solution

In spherical coordinates (r,ϕ,θ), where θ is the angle from the polar axis, the divergence of a vector function is

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla\cdot\mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla\cdot\mathbf{v}\,dV = \oint\!\!\oint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S}$$

Part (a)

If $\mathbf{v} = (1/r)\hat{\mathbf{r}}$ and D represents the sphere of radius R centered at the origin, then the left side evaluates to

$$\begin{aligned}
\iiint_D \nabla\cdot\mathbf{v}\,dV &= \int_0^\pi\int_0^{2\pi}\int_0^R \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r}\right)\right] r^2\sin\theta\,dr\,d\phi\,d\theta \\
&= \int_0^\pi\int_0^{2\pi}\int_0^R \left[\frac{1}{r^2}(1)\right] r^2\sin\theta\,dr\,d\phi\,d\theta \\
&= \left(\int_0^\pi \sin\theta\,d\theta\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_0^R dr\right) \\
&= (2)(2\pi)(R) = 4\pi R,
\end{aligned}$$

and the right side evaluates to

$$\begin{aligned}
\oint\!\!\oint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S} &= \int_0^\pi\int_0^{2\pi} \left(\frac{1}{r}\hat{\mathbf{r}}\right)\bigg|_{r=R}\cdot(\hat{\mathbf{r}}R^2\sin\theta\,d\phi\,d\theta) \\
&= \int_0^\pi\int_0^{2\pi} \left(\frac{1}{R}\hat{\mathbf{r}}\right)\cdot(\hat{\mathbf{r}}R^2\sin\theta\,d\phi\,d\theta) \\
&= R\int_0^\pi\int_0^{2\pi} \sin\theta\,d\phi\,d\theta \\
&= 4\pi R.
\end{aligned}$$

Since both sides of the divergence theorem are the same, there's no need for a delta function (apparently, $\mathbf{v}$ doesn't blow up fast enough as $r \to 0$). The divergence of $\hat{\mathbf{r}}/r$ is the quantity in square brackets.

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r} = \frac{1}{r^2}$$

If $\mathbf{v} = r^n \hat{\mathbf{r}}$ and $n < -2$, then the volume integal blows up because the integrand is singular at $r = 0$. See part (b) of Problem 1.39 for $n = 2$. If $\mathbf{v} = r^n \hat{\mathbf{r}}$ and $n > -2$, then

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r^n) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(0 \cdot \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0) \\
&= \frac{1}{r^2}\frac{d}{dr}(r^{n+2}) \\
&= \frac{1}{r^2}(n+2)r^{n+1} \\
&= (n+2)r^{n-1}.
\end{aligned}$$

Part (b)

In spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis, the curl of a vector function is

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}.$$

If $\mathbf{v} = r^n \hat{\mathbf{r}}$, then

$$\begin{aligned}
\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(0 \cdot \sin\theta) - \frac{\partial}{\partial\phi}(0)\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r^n) - \frac{\partial}{\partial r}(r \cdot 0)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r \cdot 0) - \frac{\partial}{\partial\theta}(r^n)\right]\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r\sin\theta}[(0) - (0)]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}(0) - (0)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}[(0) - (0)]\hat{\boldsymbol{\phi}} \\
&= \mathbf{0}.
\end{aligned}$$

According to part (b) of Problem 1.61,

$$\iiint_D (\nabla \times \mathbf{v})\, dV = -\oiint_{\text{bdy } D} \mathbf{v} \times d\mathbf{S}.$$

Let D be the sphere of radius R centered at the origin. The outward normal unit vector is then $\hat{\mathbf{r}}$.

$$\begin{aligned}
\int_0^\pi \int_0^{2\pi} \int_0^R [\nabla \times (r^n \hat{\mathbf{r}})]\,(r^2 \sin\theta\, dr\, d\phi\, d\theta) &= -\int_0^\pi \int_0^{2\pi} (r^n \hat{\mathbf{r}})\Big|_{r=R} \times (\hat{\mathbf{r}} R^2 \sin\theta\, d\phi\, d\theta) \\
\int_0^\pi \int_0^{2\pi} \int_0^R (\mathbf{0})\,(r^2 \sin\theta\, dr\, d\phi\, d\theta) &= -\int_0^\pi \int_0^{2\pi} (R^n \hat{\mathbf{r}}) \times (\hat{\mathbf{r}} R^2 \sin\theta\, d\phi\, d\theta) \\
\mathbf{0} &= -\int_0^\pi \int_0^{2\pi} (\mathbf{0})\, d\phi\, d\theta \\
\mathbf{0} &= \mathbf{0}
\end{aligned}$$